Biased random walks in complex networks: The role of local navigation rules

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(Received 23 September 2008; revised manuscript received 22 May 2009; published 13 July 2009)

We study the biased random-walk process in random uncorrelated networks with arbitrary degree distributions. In our model, the bias is defined by the preferential transition probability, which, in recent years, has been commonly used to study the efficiency of different routing protocols in communication networks. We derive exact expressions for the stationary occupation probability and for the mean transit time between two nodes. The effect of the cyclic search on transit times is also explored. Results presented in this paper provide the basis for a theoretical treatment of transport-related problems in complex networks, including quantitative estimation of the critical value of the packet generation rate.

DOI: 10.1103/PhysRevE.80.016107 PACS number(s): 89.75.Hc, 05.40.Fb, 05.60.−k

The problem of wandering at random in a network (or lattice) finds applications in virtually all sciences [1,2]. With only minor adjustments random walks may represent the thermal motion of electrons in a metal, or the migration of holes in a semiconductor. The continuum limit of the random-walk model is known as diffusion. It can describe the Brownian motion of a particle immersed in a fluid, as well as heat propagation, bacterial motion, and even fluctuations in the stock market. Recently, the concept of random walks has also been applied to exploring traffic in complex networks. The spectrum of network related problems includes, among many others, ordinary traffic in a city, the distribution of goods and wealth in economies, biochemical and gene expression pathways, and search (or routing) strategies on the Internet or other communication networks [3–9].

In this paper, we investigate biased random walks in complex networks, and we explore the effects of different local navigation rules on the mean first-passage (or transit) time between any pair of nodes [10]. The biased random-walk model defined in scale-free networks is particularly interesting since it has been considered as a mechanism of transport and search in real networks, including the Internet. For a long time, it was believed that the most optimum transport-related processes are based on the shortest paths between the two nodes under consideration. At the moment, however, it is understood that such a routing strategy would require a global knowledge of network topology, which is often not available. Moreover, routing strategies based on shortest paths may create inconvenient queue congestions in scale-free networks, given that the majority of the shortest paths pass through hub nodes in such structures. A possible alternative that has been suggested is to consider local navigation rules instead of global knowledge. As a consequence, a number of adequate models have been proposed (see, e.g., [5,8]). In general, the models mimic traffic in complex networks by introducing packet (particle) generation rates, as well as assigning a randomly selected source and a random destination to each packet. In these models, a common observation is that the traffic exhibits a continuous phase transition from free flow to the congested phase as a function of the packet generation rate. In the free flow state, the numbers of created and delivered particles are balanced while in the jammed state, the number of packets accumulated in the network increases with time. In this paper, we show that the random-walk model, although very simple, correctly describes the properties of the proposed traffic models in the free flow state. We calculate the transit times characterizing this state. We also provide some suggestions on how to calculate the critical packet generation rate.

Technically, we define our random walks as follows. We consider random uncorrelated networks with given node degree distributions $P(k)$ [11]. The networks are also known as random graphs or as the configuration model, and they have been repeatedly shown to be very useful in modeling different phenomena taking place in networks. We assume that the networks are connected, i.e., there exists a path between each pair of nodes. Given the graph structure, the diffusing particle (packet) is created at a randomly selected node, and it is assigned a random destination node. In the next time steps the particle passes from a node to one of its neighbors being directed by local navigation rules. In practice, it means that, being at a certain node i, the random walker performs a local search in its neighborhood (up to the first, second, or further orders) looking to see if the destination node is within the search area. If the destination is found, the particle is delivered directly to the target, following the shortest path (this rule is known as the cyclic search [5]). Otherwise, the particle continues a biased random walk, i.e., the next position (a node j) is chosen according to the prescribed probability w_{ij}.

In the following, to explore transit times characterizing biased random walks in uncorrelated networks with arbitrary degree distributions $P(k)$, we partially reproduce and generalize standard calculations for the mean first-passage time in periodic lattices [12]. At the beginning, we work out some general concepts related to biased random walks without the cyclic search. In particular, we calculate the stationary occupation probability P_i^∞ for the diffusing particle, which describes the probability that the particle is located at the node i in the infinite time limit. Then, performing simple textbook calculations, we derive formulas for the mean transit time between any pair of nodes (we would like to stress that some time ago similar calculations were done for unbiased random walks in complex networks [13]; results presented in our paper encompass the results of Ref. [13] as a special case).

The effect of the cyclic search on transit times is explored via a simple renormalization trick applied to the degrees of the nodes.

Thus, let us consider a particle that hops at discrete times between neighboring nodes of a random network described by the adjacency matrix **A**. Let $P_{ij}(t)$ be the probability that the particle starting at site i at time $t=0$ is at site j at time t. The evolution of this occupation probability is given by the master equation

$$P_{ij}(t+1) = \sum_{l=1}^{N} A_{lj} w_{lj} P_{il}(t), \tag{1}$$

where the meaning of w_{lj} has already been explained, and A_{lj} represents element of the adjacency matrix, which is equal to one if there exists a link between l and j, and zero otherwise. In the rest of the paper we perform a detailed analysis of the local navigation rules defined by the preferential transition probability [8,14]

$$w_{lj} = \frac{k_j^{\alpha}}{\sum_{m=1}^{k_l} k_m^{\alpha}}, \tag{2}$$

where the sum in the denominator runs over the neighbors of the node l, and the exponent α is the model free parameter. Note that according to formula (2) the transition probability from l to j in our biased random walk depends only on the connectivity of the next-step node j. Note also that for $\alpha=0$ we recover the ordinary unbiased random walk studied by Noh and Rieger [13].

In order to calculate the stationary occupation probability P_i^{∞}, characterizing the biased random walks studied, we average master equation (1) over the ensemble of the networks considered (i.e., we apply mean-field approximation to this equation)

$$P_j^{\infty} \simeq \sum_{l=1}^{N} \langle A_{lj}\rangle \langle w_{lj}\rangle P_l^{\infty}. \tag{3}$$

Now, before we proceed further, let us recall a few structural properties of uncorrelated networks with a given node degree distribution. First, one can show that probability of a link between any pair of nodes l and j with degrees, respectively, equal to k_l and k_j is given by (see Eq. (27) in [15])

$$\langle A_{lj}\rangle = \frac{k_l k_j}{\langle k\rangle N}. \tag{4}$$

Expression (4) is the so-called annealed network approximation. It means that a given complex network is replaced by a weighted fully connected graph. One has to be aware that this approximation gives good results for vertices with large degrees [16,17]. Second, since in uncorrelated networks the node degree distribution $Q(k_m/k_l)$ of the nearest neighbors of a node l does not depend on k_l (compare Eq. (1) in [18], and Eq. (4) in [19])

$$Q(k_m/k_l) = \frac{k_m}{\langle k\rangle} P(k_m), \tag{5}$$

the normalization factor in formula (2) is equal to

$$\sum_{m=1}^{k_l} k_m^{\alpha} = k_l \sum_{m=1}^{k_l} k_m^{\alpha} Q(k_m/k_l) = \frac{\langle k^{\alpha+1}\rangle}{\langle k\rangle} k_l, \tag{6}$$

and the transition probability w_{lj} between l and j averaged over different network realizations may be written as

$$\langle w_{lj}\rangle = \frac{\langle k\rangle}{\langle k^{\alpha+1}\rangle k_l} k_j^{\alpha}. \tag{7}$$

Finally, inserting relations (4) and (7) into simplified master equation (3), after some algebra, one obtains

$$P_j^{\infty} = \frac{k_j^{\alpha+1}}{N\langle k^{\alpha+1}\rangle}. \tag{8}$$

Note that, for $\alpha=0$, which stands for the unbiased random walk, the stationary distribution is, up to normalization, equal to the degree of the node j, i.e., $P_j^{\infty} \sim k_j$. This means that the more links a node has, the more often it will be visited by a random walker. Note also that for $\alpha=-1$, which represents the antipreferential transition probability $w_{lj} \sim 1/k_j$, the stationary occupation probability is uniform $P_j^{\infty} = 1/N$.

To test the validity of Eq. (8) we numerically calculated the fraction of random walkers found in nodes with a given connectivity k_j. The expected power-law relation $P_j^{\infty} \sim k_j^{\alpha+1}$ was found in all the α cases and for different classes of the analyzed networks [i.e., classical random graphs, and scale-free networks $P(k) \sim k^{-\gamma}$ with the characteristic exponent $\gamma=3$]; see Fig. 1. The same scaling behavior was found in Ref. [8] for the number of packets moving simultaneously in Barabási-Albert (BA) networks [20] in the free flow state. In that paper, a packet routing strategy based on the preferential transition probability [Eq. (2)], and the so-called path iteration avoidance, which means that no link can be visited twice by the same packet, was considered. At each time step R packets were generated in the network, and a fixed node capacity C, which is the number of packets a node can forward to other nodes, was assumed. The fact that our results coincide with those of Wang *et al.* [8] shows that packets may be considered as noninteracting particles (i.e., independent biased random walkers) in the free flow stationary state. One can also show that this approach can be used to estimate the critical value of packet generation rate R_c [21], as the parameter should fulfill a kind of balance equation between the node's processing efficiency C, and the number of delivered packets delivered $P_j^{\infty} R_c \langle T_{ij}\rangle$, where $\langle T_{ij}\rangle$ stands for the mean first-passage time [Eq. (13)] averaged over all pair of nodes, and, respectively, $R_c\langle T_{ij}\rangle$ corresponds to the total number of packets distributed over the whole network.

The first-passage probability $F_{ij}(t)$, namely, the probability that the random walk starting at the node i visits j for the first time at time t satisfies the well-known convolution relation [10,13]

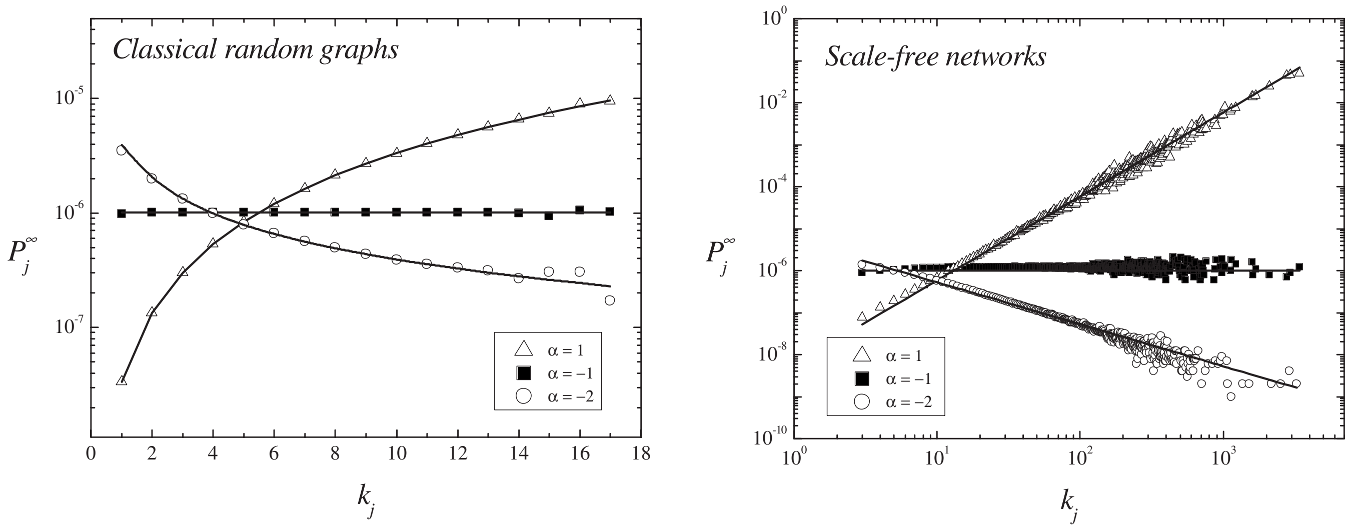

FIG. 1. Stationary probability distributions $P_j^\infty(k)$ calculated for different values of the parameter α in classical random graphs and scale-free networks. Solid lines correspond to the theoretical prediction of Eq. (8). In the case of classical random graphs, $\langle k\rangle=5$ was assumed. In scale-free networks $\gamma=3$ and $\langle k\rangle=6$ were chosen.

$$P_{ij}(t)=\delta_{t0}\delta_{ij}+\sum_{\tau=0}^{t}P_{jj}(t-\tau)F_{ij}(\tau). \tag{9}$$

The delta function term in Eq. (9) accounts for the initial condition that the walk starts at $i=j$. Applying the Laplace transform, defined as $\tilde{f}(s)=\Sigma_{t=0}^{\infty}e^{-st}f(t)$, to this equation leads to the fundamental expression

$$\tilde{F}_{ij}(s)=\frac{\tilde{P}_{ij}(s)-\delta_{ij}}{\tilde{P}_{jj}(s)}, \tag{10}$$

in which the Laplace transform of the first-passage probability $\tilde{F}_{ij}(s)$ is determined by the corresponding transform of the probability distribution $\tilde{P}_{ij}(s)$. Consequently, due to the fact that all moments

$$R_{ij}^{(n)}=\sum_{t=0}^{\infty}t^n[P_{ij}(t)-P_j^\infty] \tag{11}$$

of the exponentially decaying part of $P_{ij}(t)$ are finite, expanding $\tilde{P}_{ij}(s)$ as a power series in s

$$\tilde{P}_{ij}(s)=\frac{P_j^\infty}{1-e^{-s}}+\sum_{n=0}^{\infty}(-1)^nR_{ij}^{(n)}\frac{s^n}{n!}, \tag{12}$$

and then inserting Eq. (12) into Eq. (10), and again expanding the result as a series in s, one finally obtains the formula for the mean transit time between i and j

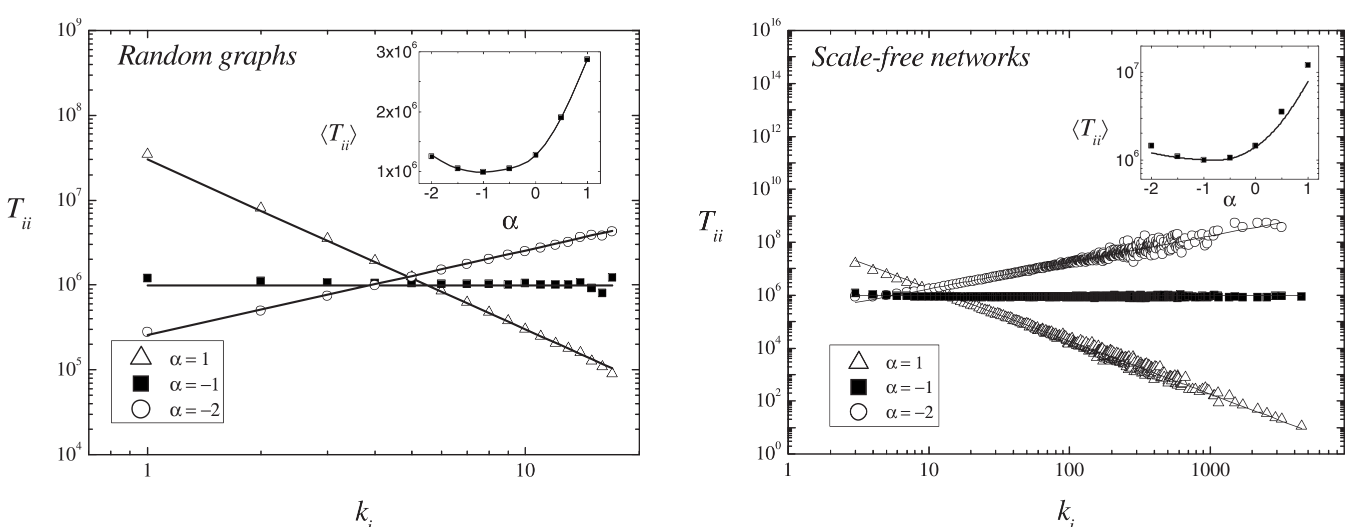

FIG. 2. Mean first return time T_{ii} vs node degree k_i (main panels), and $\langle T_{ii}\rangle$ vs α (insets) in classical random graphs ($\langle k\rangle=5$) and scale-free networks ($\gamma=3$, $\langle k\rangle=6$).

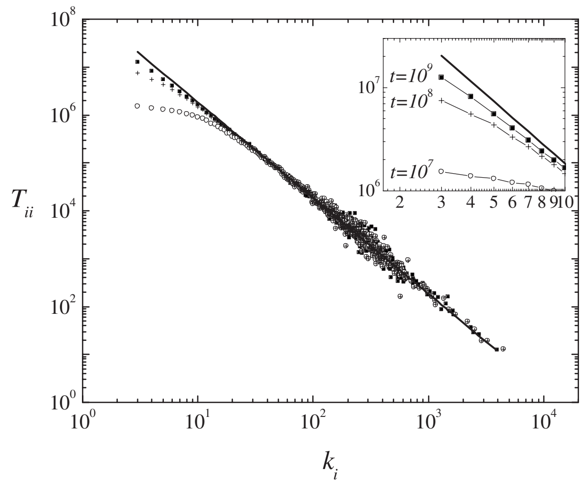

FIG. 3. First return time T_{ii} vs node degree k_i for different values of the particle's wandering time t in scale-free networks (γ=3, $\langle k\rangle$=6, α=1). Increasing the simulation time t improves agreement with the theoretical line given by Eq. (15).

$$T_{ij} = \sum_{t=0}^{\infty} tF_{ij}(t) = -\widetilde{F}'_{ij}(0) = \begin{cases} 1/P_j^\infty, & \text{for } j=i \\ [R_{jj}^{(0)} - R_{ij}^{(0)}]/P_j^\infty, & \text{for } j \neq i \end{cases} . \quad (13)$$

Recall that P_j^∞ [Eq. (8)] corresponds to the stationary occupation probability, which has already been calculated.

Figure 2 shows how the mean first return time T_{ii} of the biased diffusing particle wandering in random network depends on k_i. In the figure, numerically calculated transit times are indicated by scattered points, whereas their values predicted by theory (13), namely,

$$T_{ii} = \frac{N\langle k^{\alpha+1}\rangle}{k_i^{\alpha+1}}, \quad (14)$$

are represented by solid lines. Subsets given in the figure show how the mean first return time $\langle T_{ii}\rangle$ averaged over all nodes depends on α (i.e., on local navigation rules governing the diffusing particle)

$$\langle T_{ii}\rangle = N\langle k^{\alpha+1}\rangle\langle k^{-\alpha-1}\rangle, \quad (15)$$

and they indirectly show how fast the biased random walk is. The discrepancies between numerical simulations and their theoretical prediction given by Eq. (15) shown in the subset result from the limited wandering time of a diffusing particle, cf. Fig. 3. The minimum value of $\langle T_{ii}\rangle$ observed for $\alpha_m \simeq -1$ in classical random graphs indicates that the antipreferential transition probability [Eq. (2)] causes the slowest exploration of the networks considered, which, in turn, causes, in the case of such a navigation rule, the relaxation part of the occupation probability $P_{ii}(t) - P_i^\infty$ to converge to zero much more slowly than in the case of other values of the parameter α (the same reasoning applies to the case of $\alpha_m \simeq -0.5$ in scale-free networks). This reasoning implies that, although in general the parameters $R_{jj}^{(0)}$ and $R_{ij}^{(0)}$ in formula (13) for the mean transit time T_{ij} cannot be easily calculated, the expected fast convergence of the occupation probability $P_{ij}(t)$ toward the stationary distribution P_j^∞ for $|\alpha - \alpha_m| \neq 0$ allows one to simplify the relation

$$T_{ij} \simeq \left(\sum_{t=0}^{2} [P_{jj}(t) - P_j^\infty] - \sum_{t=0}^{0} [P_{ij}(t) - P_j^\infty] \right) / P_j^\infty$$
$$= \frac{N\langle k^{\alpha+1}\rangle}{k_j^{\alpha+1}} + \frac{N\langle k\rangle^2}{\langle k^2\rangle} \frac{1}{k_j} - 2, \quad (16)$$

where $P_{ij}(0)=0$, $P_{jj}(0)=1$, $P_{jj}(1)=0$, and $P_{jj}(2)=\langle w_{lj}\rangle$ [Eq. (7)] with $k_l = \langle k^2\rangle / \langle k\rangle$ standing for the average degree of nearest neighbors. In Fig. 4 one can see that the theoretical prediction of Eq. (16) agrees quite well with the numerical calculations of T_{ij}. As expected, approximate formula (16) works better for the parameter $\alpha > \alpha_m$. We have also checked that the mean first-passage time T_{ij} between any pair of nodes does not depend on the source node i in the networks considered. It only depends on the destination node j.

Knowing the mean transit time [Eq. (13)] of the biased random walk, the effect of the cyclic search on the quantity can be calculated through a simple trick that consists of dividing the walk between any pair of nodes i and j into two parts: the first part is before the diffusing particle arrives in

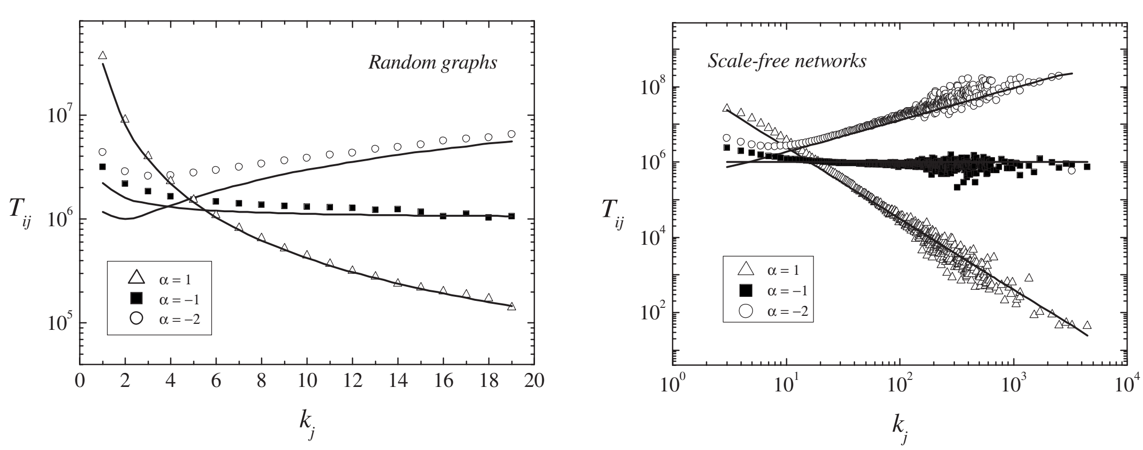

FIG. 4. Mean transit time T_{ij} between two nodes i and j vs connectivity of the target node k_j in classical random graphs ($\langle k\rangle$=5) and scale-free networks (γ=3, $\langle k\rangle$=6).

the neighborhood of the target, and the second part is when the particle sees its destination and follows the shortest path. Distinguishing between the two parts allows us to treat the first part as an ordinary biased random walk from a node i to an arbitrary node in the distance x from j. The stationary occupation probability for these nodes taken together, $P_j^\infty(x)$, is equal to the sum of probabilities representing the separate nodes that belong to the group [note that $P_j^\infty(0) \equiv P_j^\infty$, Eq. (8)]. In the case of $x=1$, the sum runs over the k_j nearest neighbors of j. Since the average connectivity of a nearest neighbor is $\langle k^2 \rangle / \langle k \rangle$ [from Eq. (5)], thus, due to Eq. (8), occupation probability for the nearest neighborhood is given by

$$P_j^\infty(1) = \frac{k_j}{N\langle k^{\alpha+1}\rangle}\left(\frac{\langle k^2\rangle}{\langle k\rangle}\right)^{\alpha+1}, \tag{17}$$

and in general one can show that

$$P_j^\infty(x) \propto k_j. \tag{18}$$

With the stationary distribution $P_j^\infty(x)$, the mean first-passage time $T_{ij}(x)$ in the cyclic search problem can be calculated from the formula below:

$$T_{ij}(x) = x + T_{iJ} = x + [R_{JJ}^{(0)} - R_{iJ}^{(0)}]/P_j^\infty(x) \propto \frac{1}{P_j^\infty(x)} = \frac{1}{k_j}, \tag{19}$$

where J stands for the xth nearest neighborhood of j. The parameters $R_{JJ}^{(0)}$ and $R_{iJ}^{(0)}$ are given by Eq. (11), with J corresponding to the group of nodes that screens the destination node j. Of course, in the case of $x=1$, these nodes correspond to the nearest neighborhood, and $P_{JJ}(t)$, appearing in the formula for $R_{JJ}^{(0)}$, describes the probability that a diffusing particle starting at J at time $t=0$ is at J at time t. In Fig. 5, one can see that scaling relation (19) for the mean transit time in the cyclic search problem with $x \geq 1$ really does work.

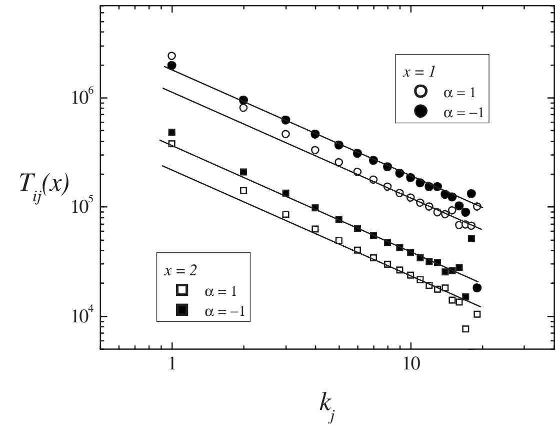

FIG. 5. Mean transit time $T_{ij}(x)$ for the cyclic search problem in classical random graphs with $\langle k \rangle=5$. The solid lines have the slop predicted by Eq. (19), i.e., $T_{ij}(x) \propto k_j^{-1}$.

In summary, we have studied the biased random-walk process in random uncorrelated networks with arbitrary degree distributions. In our model, the bias was defined by the preferential transition probability [Eq. (2)] (see also another paper on biased diffusion in random networks [22]). We have calculated the expression for the stationary occupation probability, and we have derived formulas for the mean first-passage times between any pair of nodes. The effect of the cyclic search on transit times was also explored. We have also shown that the random-walk approach can be used to explain some properties of traffic dynamics in communication networks. Other traffic-related problems that can be solved using this approach include, among many others, the microscopic explanation of the phase transition from free flow to the jammed phase and the quantitative estimation of the critical value of the packet generation rate in scale-free networks [8]. We leave these problems for our future work [21].

The work was funded in part by the State Committee for Scientific Research in Poland under Grant No. 1P03B04727 (A.F.), and the European Commission Project No. CREEN FP6-2003-NEST-Path-012864 (P.F.).

[1] R. M. Mazo, *Brownian Motion: Fluctuations, Dynamics, and Applications* (Oxford University Press, New York, 2002).
[2] D. ben-Avraham and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems* (Cambridge University Press, New York, 2000).
[3] L. A. Adamic, R. M. Lukose, A. R. Puniyani, and B. A. Huberman, Phys. Rev. E **64**, 046135 (2001).
[4] B. Tadic and G. J. Rodgers, Adv. Complex Syst. **5**, 445 (2002).
[5] B. Tadic and S. Thurner, Physica A **332**, 566 (2004).
[6] B. J. Kim, C. N. Yoon, S. K. Han, and H. Jeong, Phys. Rev. E **65**, 027103 (2002).
[7] M. Rosvall, P. Minnhagen, and K. Sneppen, Phys. Rev. E **71**, 066111 (2005).
[8] W. X. Wang, B. H. Wang, C. Y. Yin, Y. B. Xie, and T. Zhou, Phys. Rev. E **73**, 026111 (2006).
[9] R. Germano and A. P. S. de Moura, Phys. Rev. E **74**, 036117 (2006).
[10] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, New York, 2001).
[11] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[12] B. D. Hughes, *Random Walks and Random Environments* (Clarendon Press, Oxford, 1995), Vol. 1.
[13] J. D. Noh and H. Rieger, Phys. Rev. Lett. **92**, 118701 (2004).
[14] S.-J. Yang, Phys. Rev. E **71**, 016107 (2005).
[15] A. Fronczak and P. Fronczak, Phys. Rev. E **74**, 026121 (2006).
[16] G. Bianconi, Phys. Lett. A **303**, 166 (2002).
[17] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008).

[18] A. Fronczak, P. Fronczak, and J. A. Hołyst, AIP Conf. Proc. **776**, 52 (2005).
[19] M. E. J. Newman, in *Handbook of Graphs and Networks*, edited by S. Bornholdt and H. G. Schuster (Wiley-VCH, Berlin, 2003).
[20] A. L. Barabási, R. Albert, and H. Jeong, Physica A **272**, 173 (1999).
[21] P. Fronczak and A. Fronczak (unpublished).
[22] V. Sood and P. Grassberger, Phys. Rev. Lett. **99**, 098701 (2007).